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Generalizations of Kannan-type Mappings and Their Fixed Point Theorems in Modular $b-$Metric Spaces

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Abstract

This paper establishes fixed point theorems for generalized Kannan-type mappings in modular $b-$metric spaces by using subadditive altering distance functions. The fixed point theorems presented in this paper extend the existing results for generalized Kannan-type mappings in $b-$metric spaces by adding the Δ_2-type condition into the modular $b-$metric space framework.

Keywords: Δ_2-condition; fixed point; Kannan-type mapping; modular $b-$metric space.

1 Introduction

In the past decade, the advancement of fixed point theorems has progressed rapidly, particularly the advancement of the Banach Contraction Principle(BCP for short), which provides a fixed point theorem for contraction mappings in metric spaces [4]. Let $\mathfrak{M}$ be an arbitrary nonempty set and an element $u \in \mathfrak{M}$ is said to be a fixed point(briefly FP) of self-mapping Γ on $\mathfrak{M}$ if $\Gamma u = u$ [1]. The generalization of metric space is one of the main factors supporting the development of fixed point theory. Numerous generalizations of metric spaces have been introduced over the years. Among the most notable are the $b-$metric space, introduced Czerwik [12], the cone metric space developed by Long-Guang and Xian [22] and the modular metric space introduced by Chistyakov [9].

Building upon the $b-$metric space framework, several further extensions have emerged, including the vector-valued $b-$metric space, introduced by Boriceanu in 2009 [5]; the $b-$fuzzy metric space [31], the partial $b-$metric space [33] and the modular $b-$metric space (briefly MbMS) [13]. In the setting of cone metric spaces, Shaddad et al. [32] established several FP results for multivalued mappings. Within the framework of partial $b-$metric spaces, Anwar et al. [3] developed FP theorems for contraction and almost contraction mappings were established. Additionally, Nazam et al. [25] presented common FP theorems for the mappings in the same space. In the setting of $b-$fuzzy metric spaces, Saleem et al. [30] introduced approximate FP results for generalized type fuzzy contractive mappings. More recently, further advancements in vector-valued $b-$metric spaces were discussed by Nazam et al. [26], who proposed generalized F-contraction mappings with corresponding FP theorems.

Another factor contributing to the development of fixed point theory is the emergence of various generalizations of contraction mappings. In 1969, Kannan [21] introduced a notable generalization of contraction mappings in metric spaces by removing the requirement of continuity and established a FP theorem that extends the BCP. This theorem later became known as the Kannan FP theorem. More recently, Mudhesh et al. [23] developed FP theorems for integral Khan-type multivalued $(\psi, \varphi)-$contractions in complete metric spaces. In the broader context of $b-$metric spaces, Ali et al. [2] proposed generalized FP theorems via nonlinear θ_b-contractions. Additionally, based on the work of Kannan [21], Haokip and Goswami [17] presented FP theorems for generalized Kannan-type(briefly K–t) mappings with subadditive altering distance functions in $b-$metric spaces. In the setting of MbMS, Ozturk et al. [27] characterized a new class of simulation functions and derived corresponding common FP theorems. Furthermore, in the framework of generalized MbMS, Hayati et al. [18] introduced a FP theorem for $(\psi, \varphi)_\Omega-$contraction mapping in the extended MbMS, meanwhile Tyagi et al. [36] introduced the concept of F-modular $b-$metric spaces and established generalizations of several well-known FP theorems, including the BCP and the Kannan FP theorems.

In addition to the development of fixed point theorems, various applications of these theorems have also been explored, one of which is presented in [20], where a FP theorem is applied to an optimization problem in the setting of $b-$metric spaces. Despite the advancements, several FP theorems for K–t mappings have not yet been presented, particularly those using subadditive altering distance function in MbMS. Therefore, in this work, we study about FP theorems for generalized K–t mappings in MbMS.

2 Related Works

Kannan [21] established the following FP theorems.

Theorem 2.1. *Let $(\mathfrak{M}, \varrho)$ be an arbitrary complete metric space and let Γ be a self-mapping on $\mathfrak{M}$ such that there exists $\mathfrak{p} \in \left[0, \frac{1}{2}\right)$ satisfying,*

$$\varrho(\Gamma u, \Gamma v) \leq \mathfrak{p}[\varrho(u, \Gamma u) + \varrho(v, \Gamma v)], \qquad \text{for all } u, v \in \mathfrak{M}. \tag{1}$$

Then, Γ has a unique FP in $\mathfrak{M}$.

In 1993, Czerwik [12] introduced a generalization of metric spaces, called $b-$metric spaces, which are defined as follows.

Definition 2.1. *Let $\mathfrak{M}$ be an arbitrary nonempty set. A function $\varrho : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ is called a $b-$metric space if there exists $\mathfrak{s} \geq 1$ such that the following axioms hold for all $u, v, w \in \mathfrak{M}$:*

- (M1) $\varrho(u, v) = 0 \Leftrightarrow u = v$.
- (M2) $\varrho(u, v) = \varrho(u, v)$.
- (M3) $\varrho(u, v) \leq \mathfrak{s}[\varrho(u, w) + \varrho(w, v)]$.

A pair $(\mathfrak{M}, \varrho)$ is called a $b-$metric space. It is clear that every metric space is a $b-$metric space with $\mathfrak{s} = 1$.

Based on FP theorems for K–t mappings in metric spaces introduced by Górnicki [16], Haokip and Goswami extended these results to $b-$metric spaces by employing subadditive altering distance functions. Faraji and Nourozi [14] provided the definition of altering distance functions, while Haokip and Goswami [17] introduced the concept of subadditive altering distance functions.

Definition 2.2. *A function $\mho : [0, \infty) \to [0, \infty)$ is called subadditive altering distance function if it satisfies the following conditions:*

- (i) $\mho$ *is continuous.*
- (ii) $\mho$ *is strictly increasing.*
- (iii) $\mho(\mathfrak{t}) = 0 \Leftrightarrow \mathfrak{t} = 0$.
- (iv) $\mho(\mathfrak{s} + \mathfrak{t}) \leq \mho(\mathfrak{s}) + \mho(\mathfrak{t})$ *for all* $\mathfrak{s}, \mathfrak{t} \in [0, \infty)$.

In 1999, Corazza [11] established a property of subadditive functions, which is stated as follows;

Proposition 2.1. *If a function $\mho : [0, \infty) \to [0, \infty)$ is subadditive, then for all $\mathfrak{t} \in [0, \infty)$ and $n \in \mathbb{N}$, the following condition holds,*

$$\mho(n\mathfrak{t}) \leq n\, \mho(\mathfrak{t}). \tag{2}$$

In 1959, Musielak and Orlicz [24] introduced a functional called a modular, defined on a linear space. Later, Chistyakov [9] introduced the concept of a modular metric space, which generalizes metric spaces in the context of modular spaces. According to these developments, Ege and Alaca [13] proposed a further generalization in 2018, introducing the concept of a MbMS, which extends both $b-$metric spaces and modular metric spaces.

Definition 2.3. *Let $\mathfrak{M}$ be an arbitrary nonempty set and $\mathfrak{s} \geq 1$. A function $\wp : (0, \infty) \times \mathfrak{M} \times \mathfrak{M} \to [0, \infty]$ is called a modular $b-$metric(MbM for short) if the following axioms hold for all $u, v, w \in \mathfrak{M}$:*

- $(B1)$ $\wp_\lambda(u, v) = 0$ *for all* $\lambda > 0 \Leftrightarrow u = v$.
- $(B2)$ $\wp_\lambda(u, v) = \wp_\lambda(v, u)$ *for all* $\lambda > 0$.
- $(B3)$ $\wp_{\lambda+\mu}(u, v) \leq s[\wp_\lambda(u, w) + \wp_\mu(w, v)]$ *for all* $\lambda, \mu > 0$.

The pair $(\mathfrak{M}, \wp)$ is called a modular $b-$metric space(briefly MbMS). If Axiom $(B1)$ is replaced with the following condition,

$$\wp_\lambda(u, u) = 0, \qquad \text{for all } u \in \mathfrak{M},$$

then, $\wp$ is called a pseudomodular $b-$metric on $\mathfrak{M}$. It is clear that every MbM $\wp$ is a pseudomodular $b-$metric. Furthermore, if $\wp$ is a pseudomodular $b-$metric, then for all $0 < \mu < \lambda$ and $u, v \in \mathfrak{M}$, we have

$$\wp_\lambda(u, v) = \wp_{\lambda-\mu+\mu}(u, v) \leq \mathfrak{s}[\wp_{\lambda-\mu}(u, u) + \wp_\mu(u, v)] = \mathfrak{s}\wp_\mu(u, v). \tag{3}$$

In addition to defining MbMS, Ege and Alaca [13] also introduced a definition of a binary relation on $\mathfrak{M}$. However, in this paper, we provide an alternative definition of a binary relation on $\mathfrak{M}$ based on the work in [10].

Definition 2.4. *Let $(\mathfrak{M}, \wp)$ be an arbitrary MbMS. The binary relation $\overset{\wp}{\sim}$ on $u, v \in \mathfrak{M}$ is defined by,*

$$u \overset{\wp}{\sim} v \Leftrightarrow \wp_\lambda(u, v) < \infty, \qquad \text{for some } \lambda > 0.$$

For a fixed $u^0 \in \mathfrak{M}$, the modular set is defined as,

$$\mathfrak{M}^*_\wp = \mathfrak{M}^*_\wp(u^0) = \{u \in \mathfrak{M} : \exists \lambda = \lambda(u) > 0, \quad \text{such that } \wp_\lambda(u, u^0) < \infty\}.$$

Furthermore, we define the set,

$$\mathfrak{M}_\wp = \mathfrak{M}_\wp(u^0) = \{u \in \mathfrak{M} : \wp_\lambda(u, u^0) \to 0, \quad \text{as } \lambda \to \infty\}.$$

It is easy to show that the binary relation $\overset{\wp}{\sim}$ is an equivalence relation dan $\mathfrak{M}_\wp \subseteq \mathfrak{M}^*_\wp$. Moreover, $\mathfrak{M}^*_\wp$, when equipped with $\wp$, forms a MbMS. In summary, $\mathfrak{M}^*_\wp$ is referred to as a MbMS.

Some properties of subadditive functions are used to prove the FP theorem. The following property of subadditive functions is derived from [17, 19].

Proposition 2.2. *Let $\mathfrak{M}^*_\wp$ be a MbMS and let $\lambda > 0$. If $\mho$ is subadditive, then for all $\mathfrak{p} \in [0, 1)$ and $u, v \in \mathfrak{M}^*_\wp$,*

$$\mho(\wp_\lambda(u, v)) \leq \mathfrak{p}\, \mho(\wp_\lambda(\mathfrak{a}, \mathfrak{b})), \tag{4}$$

for some $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}^*_\wp$ implies that,

$$\wp_\lambda(u, v) \leq \mathfrak{p}'\wp_\lambda(\mathfrak{a}, \mathfrak{b}), \tag{5}$$

for some $\mathfrak{p}' \in [0, 1)$.

3 Materials and Methods

The research method used in this paper is a literature study. It begins with a study of Kannan FP theorems in metric spaces, followed by an exploration of $b-$metric spaces and the FP theorems for K–t mappings. The study then progresses to modular $b-$metric spaces and converges to the development of a FP theorem for generalized K–t mappings in MbMS.

We introduce several definitions related to sequences in MbMS. These definitions are developed by extending the concepts of closed sets, convergent sequences, Cauchy sequences, and complete spaces in modular metric spaces as presented in [9, 28]. Additionally, we adopt the definition of continuous functions from [7], extend the definition of bounded sets in modular metric spaces from [34], and expand upon the concept of boundedly compact sets in $b-$metric spaces from [17]. Based on these foundations, we present the following definitions;

Definition 3.1. *Let $\mathfrak{M}^*_\wp$ be a MbMS,*

1. *A sequence $\{u_n\} \subset \mathfrak{M}^*_\wp$ is said to be $\wp-$convergent to $u \in \mathfrak{M}^*_\wp$, if there exists $\lambda > 0$, such that $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$.*
2. *A sequence $\{u_n\} \subset \mathfrak{M}^*_\wp$ is called a $\wp-$Cauchy sequence(briefly $\wp-$CS), if there exists $\lambda > 0$, such that $\wp_\lambda(u_n, u_m) \to 0$ as $m, n \to \infty$.*
3. *The MbMS $\mathfrak{M}^*_\wp$ is said to be $\wp-$complete, if every $\wp-$CS in $\mathfrak{M}^*_\wp$ is $\wp-$convergent. That is, if $\{u_n\} \subset \mathfrak{M}^*_\wp$ and there exists $\lambda > 0$, such that $\wp_\lambda(u_n, u_m) \to 0$ as $n \to \infty$, then, there exists $u \in \mathfrak{M}^*_\wp$, such that $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$.*
4. *A function $\Gamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ is said to be $\wp-$continuous if for every sequence $\{u_n\} \subset \mathfrak{M}^*_\wp$ that $\wp-$convergent to $u \in \mathfrak{M}^*_\wp$, the sequence $\{\Gamma u_n\}$ $\wp-$convergent to Γu. That is, if there exists $\lambda > 0$, such that $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$, then, $\wp_\lambda(\Gamma u_n, \Gamma u) \to 0$ as $n \to \infty$.*
5. *A set $C \subset \mathfrak{M}^*_\wp$ is said to be $\wp-$closed if every sequence $\{u_n\} \subset C$ that is $\wp-$convergent to $u \in \mathfrak{M}^*_\wp$ satisfies $u \in C$.*
6. *A set $C \subset \mathfrak{M}^*_\wp$ is $\wp-$bounded if there exist $\lambda > 0$ and $\epsilon > 0$, such that $\wp_\lambda(u, v) \leq \epsilon$ for all $u, v \in C$.*
7. *The MbMS $\mathfrak{M}^*_\wp$ is called $\wp-$boundedly compact if every $\wp-$bounded sequence in $\mathfrak{M}^*_\wp$ has a $\wp-$convergent subsequence, i.e. if $\{u_n\} \subset \mathfrak{M}^*_\wp$ satisfies that there exist $\lambda > 0$ and $\epsilon > 0$, such that $\wp_\lambda(u_n, u_m) \leq \epsilon$ for all $n, m \in \mathbb{N}$, then, there exists a subsequence $\{u_{n_k}\}$ and a $u \in \mathfrak{M}^*_\wp$, such that $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$.*

The conditions for defining convergent and Cauchy sequences presented in this paper are less restrictive than those outlined in [13]. As a result, some properties discussed in [13] do not always hold. By adopting these more lenient conditions, we aim to establish a broader framework that can encompass a wider variety of sequences. This approach may lead to new insights and results in the study of convergent and Cauchy sequences, allowing for a more flexible analysis of their properties.

In metric spaces, every convergent sequence is also a Cauchy sequence. However, in MbMS, this statement does not always hold. Chistyakov [10] provided a condition, known as the Δ_2-condition, under which the statement holds in MbMS. In this paper, we develop a more general version of the Δ_2-condition, introduced by Turkoglu and Manav [35], which extends the Δ_2-condition presented by Chistyakov [10].

Definition 3.2. *A MbM $\wp$ on a nonempty set $\mathfrak{M}^*_\wp$ is said to satisfy Δ_2-condition, if for some $\lambda > 0$, $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$ implies that $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$ for all $\lambda > 0$.*

Using the Δ_2-condition, we establish the following property that describes the connection between convergent and Cauchy sequences.

Proposition 3.1. *Let $\wp$ be a MbM on nonempty set $\mathfrak{M}^*_\wp$ satisfying the Δ_2-condition and let C be a subset of $\mathfrak{M}^*_\wp$. Then the following properties hold:*

1. *If $\{u_n\} \subset C$ is $\wp-$convergent, then $\{u_n\}$ is a $\wp-$CS in C.*
2. *If $\{u_n\} \subset C$ is $\wp-$converges to $u \in \mathfrak{M}^*_\wp$, then for all $\lambda > 0$, $\wp_\lambda(u_n, u) \to 0$ as $n \to \infty$.*
3. *If $\{u_n\} \subset C$ is a $\wp-$CS, then for all $\lambda > 0$, $\wp_\lambda(u_n, u_m) \to 0$ as $n, m \to \infty$.*

Proof. This proposition can be easily proven. □

Next, we present a property related to closed set in MbMS.

Proposition 3.2. *Let $\mathfrak{M}^*_\wp$ be a $\wp-$complete MbMS, where $\wp$ satisfies the Δ_2-condition. A subset $C \subseteq \mathfrak{M}^*_\wp$ is $\wp-$closed if and only if C is $\wp-$complete.*

Proof. First, suppose C is $\wp-$closed. We will show that C is $\wp-$complete. Let $\{u_n\} \subset C$ be an arbitrary $\wp-$CS. Since $\{u_n\}$ is also $\wp-$CS in $\mathfrak{M}^*_\wp$ and $\mathfrak{M}^*_\wp$ is $\wp-$complete, there exists $u \in \mathfrak{M}^*_\wp$ and $\lambda > 0$ such that,

$$\wp_\lambda(u_n, u) \to 0, \qquad \text{as } n \to \infty.$$

Because C is $\wp-$closed, it follows that $u \in C$. Thus, every $\wp-$CS in C converges to a point in C, proving that C is $\wp-$complete. Conversely, if C is $\wp-$complete, we will show that C is $\wp-$closed. Suppose C is not $\wp-$closed. Then there exists a sequence $\{u_n\} \subset C$ that $\wp-$converges to a point $u \in \mathfrak{M}^*_\wp$, where $u \notin C$. Thus, there exists $\lambda_0 > 0$ such that,

$$\wp_{\lambda_0}(u_n, u) \to 0, \qquad \text{as } n \to \infty.$$

Since $\wp$ satisfies the Δ_2-condition, then $\{u_n\}$ is a $\wp-$CS in C. By the completeness of C, there exists $w \in C$ and $\lambda_1 > 0$ such that,

$$\wp_{\lambda_1}(u_n, w) \to 0, \qquad \text{as } n \to \infty.$$

Again, because $\wp$ satisfies Δ_2-condition, then for all $\lambda > 0$, we have

$$\wp_\lambda(u_n, w) \to 0 \qquad \text{and} \qquad \wp_\lambda(u_n, u) \to 0, \qquad \text{as } n \to \infty.$$

Moreover, it is clear that,

$$\wp_{\frac{\lambda}{2}}(u_n, w) \to 0 \qquad \text{and} \qquad \wp_{\frac{\lambda}{2}}(u_n, u) \to 0, \qquad \text{as } n \to \infty.$$

Using the triangle inequality for all $\lambda > 0$, we have

$$\wp_\lambda(u, v) \leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u, u_n) + \wp_{\frac{\lambda}{2}}(u_n, w)\right].$$

As a result, it follows that $\wp_\lambda(u, v) \to 0$, implying $u = w$. However, since $v \in C$ and $u = w$, this implies $u \in C$, which contradicts the assumption that $u \notin C$. Hence, C must be $\wp-$closed. □

In 2021, Chaira et al. [8] introduced a Δ_2-type condition in modular metric spaces that is stronger than the Δ_2-condition [10]. Thus, we give a new type Δ_2-condition in below;

Definition 3.3. *A MbM $\wp$ is said to satisfy the Δ_2-type condition if for every $\alpha > 0$, there exists $K_\alpha > 0$, such that,*

$$\wp_{\frac{\lambda}{\alpha}}(u, v) \leq K_\alpha \wp_\lambda(u, v).$$

It is easy to show that if $\wp$ satisfies the Δ_2-type condition, then $\wp$ also satisfies the Δ_2-condition. Next, we present a property of subsets of the $\wp-$bounded space $\mathfrak{M}^*_\wp$ under the assumption that $\wp$ satisfies the Δ_2-type condition.

Proposition 3.3. *Let $\mathfrak{M}^*_\wp$ be a $\wp-$bounded MbMS where $\wp$ satisfies the Δ_2-type condition and let $C \subset \mathfrak{M}^*_\wp$. If C is $\wp-$bounded, then for all $\lambda > 0$, there exists $K_\lambda > 0$ such that,*

$$\wp_\lambda(u, v) \leq K_\lambda, \qquad \textit{for all } u, v \in \mathfrak{M}^*_\wp. \tag{6}$$

Proof. Let $\lambda > 0$ be arbitrary. Since C is $\wp-$bounded, there exist $\lambda_0 > 0$ and $\epsilon_0 > 0$ such that,

$$\wp_{\lambda_0}(u, v) \leq \epsilon_0, \qquad \text{for all } u, v \in C.$$

Given that $\wp$ satisfies the Δ_2-type condition for $\alpha = \dfrac{\lambda_0}{\lambda} > 0$, there exists $K_\alpha > 0$ such that,

$$\wp_\lambda(u, v) = \wp_{\frac{\lambda_0}{\alpha}}(u, v) \leq K_\alpha \wp_{\lambda_0}(u, v) \leq K_\alpha \epsilon_0.$$

Thus, by taking $K_\lambda = K_\alpha \epsilon_0$, we conclude that for arbitrary $\lambda > 0$, there exists $K_\lambda > 0$ such that,

$$\wp_\lambda(u, v) \leq K_\lambda, \qquad \text{for all } u, v \in C.$$

□

It is clear that the set $\{K_\lambda | \lambda > 0\}$, where K_λ satisfies Inequality (6), is not always bounded. However, in the following theorems, we require the boundedness of the set. That is,

$$\sup\{K_\lambda | \lambda > 0\} < \infty,$$

or equivalently, there exists $K > 0$ such that,

$$K_\lambda \leq K, \qquad \text{for all } \lambda > 0. \tag{7}$$

4 Results

In this paper, we present two types of theorems, specifically those that do not use a subadditive altering distance function, and those that do.

4.1 Some theorem without subadditive altering distance function

In 2017, Górnicki [16] presented a FP theorem for K–t mappings in metric spaces. In the proof of this theorem, Górnicki introduced a lemma. Based on this, we give a similar lemma in the context of MbMS.

Lemma 4.1. *Let $\mathfrak{M}^*_\wp$ be a MbMS such that $\wp$ satisfies the Δ_2-type condition. Let C be a nonempty, $\wp-$closed and $\wp-$bounded subset of $\mathfrak{M}^*_\wp$ and $\Gamma : C \to C$ be a mapping such that there exists $k \in \left[0, \frac{1}{\mathfrak{s}^2}\right)$ and $\lambda > 0$ satisfying,*

$$\wp_\lambda(\Gamma u, \Gamma v) \leq k\left[\wp_\lambda(u, \Gamma u) + \wp_\lambda(v, \Gamma v)\right], \qquad \text{for all } u, v \in \mathfrak{M}^*_\wp. \tag{8}$$

Assume there exist $a, b \in \mathbb{R}$ such that $a \in [0, 1)$ and $b \in (0, \infty)$, and the set $\{K_\lambda | \lambda > 0\}$, where K_λ satisfies Inequality (6), is bounded. Furthermore, for each $u \in C$, there exists $\mathfrak{w} \in C$ such that,

$$\wp_\lambda(\mathfrak{w}, \Gamma\mathfrak{w}) \leq a\, \wp_\lambda(u, \Gamma u) \qquad \text{and} \qquad \wp_\lambda(\mathfrak{w}, u) \leq b\, \wp_\lambda(u, \Gamma u).$$

Then, Γ has at least one FP.

Proof. Let u_0 be an arbitrary element of C. By the given assumption, there exists $u_1 \in C$ such that,

$$\wp_\lambda(u_1, \Gamma u_1) \leq a\wp_\lambda(u_0, \Gamma u_0) \qquad \text{and} \qquad \wp_\lambda(u_1, u_0) \leq b\wp_\lambda(u_0, \Gamma u_0).$$

Since $u_1 \in C$, there exists $u_2 \in C$ such that,

$$\wp_\lambda(u_2, \Gamma u_2) \leq a\wp_\lambda(u_1, \Gamma u_1) \qquad \text{and} \qquad \wp_\lambda(u_2, u_1) \leq b\wp_\lambda(u_1, \Gamma u_1).$$

Again, since $u_2 \in C$, there exists $u_3 \in C$ such that,

$$\wp_\lambda(u_3, \Gamma u_3) \leq a\wp_\lambda(u_2, \Gamma u_2) \qquad \text{and} \qquad \wp_\lambda(u_3, u_2) \leq b\wp_\lambda(u_2, \Gamma u_2).$$

If this process continues, we obtain a sequence $\{u_n\} \subset C$ such that,

$$\wp_\lambda(\Gamma u_{n+1}, u_{n+1}) \leq a\, \wp_\lambda(\Gamma u_n, u_n) \qquad \text{and} \qquad \wp_\lambda(u_{n+1}, u_n) \leq b\, \wp_\lambda(\Gamma u_n, u_n), \qquad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Thus, it follows that,

$$\wp_\lambda(\Gamma u_n, u_n) \leq a^n \wp_\lambda(\Gamma u_0, u_0), \qquad \text{for all } n \in \mathbb{N} \cup \{0\}, \tag{9}$$

and therefore,

$$\wp_\lambda(u_{n+1}, u_n) \leq b\, a^n \wp_\lambda(\Gamma u_0, u_0), \qquad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Since $a \in [0, 1)$, it follows that $a^n \to 0$ as $n \to \infty$, hence,

$$\wp_\lambda(u_{n+1}, u_n) \to 0, \qquad \text{as } n \to \infty. \tag{10}$$

Now, let $m, n \in \mathbb{N}$ be two arbitrary integers with $k = m-n$. We proceed by mathematical induction to show that for all $k \in \mathbb{N}$ with $k = m - n$, we have

$$\wp_\lambda(u_m, u_n) = \wp_\lambda(u_{n+k}, u_n) \to 0,$$

as $n, m \to \infty$. For the base case $k = 1$, it follows that $\wp_\lambda(u_m, u_n) = \wp_\lambda(u_{n+1}, u_n)$. Based on Inequality (10), we have

$$\wp_\lambda(u_m, u_n) = \wp_\lambda(u_{n+1}, u_n) \to 0, \qquad \text{as } n \to \infty.$$

Thus, the statement holds for $k=1$. Next, assume that the statement holds for $k=l$, i.e.,

$$\wp_\lambda(u_m,u_n)=\wp_\lambda(u_{n+l},u_n)\to 0 \text{ as } n\to\infty. \tag{11}$$

We now prove that the statement holds for $k=l+1$. For all $n,m\in\mathbb{N}$, we have

$$\wp_\lambda(u_m,u_n)=\wp_\lambda(u_{n+(l+1)},u_n)\le \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_{(n+l)+1},u_{n+1})+\wp_{\frac{\lambda}{2}}(u_{n+l},u_n)\right].$$

Since $\wp$ satisfies the Δ_2-type condition, it follows that $\wp$ also satisfies the Δ_2-condition. By the induction hypothesis (11) and the result from (10), we have

$$\wp_{\frac{\lambda}{2}}(u_{n+l},u_n)\to 0 \quad \text{and} \quad \wp_{\frac{\lambda}{2}}(u_{(n+l)+1},u_{n+l})\to 0, \quad \text{as } n\to\infty.$$

Therefore, we obtain

$$\wp_\lambda(u_m,u_n)\le \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_{(n+l)+1},u_{n+1})+\wp_{\frac{\lambda}{2}}(u_{n+l},u_n)\right]\to 0, \quad \text{as } n\to\infty.$$

Thus, $\{u_n\}$ is $\wp-$CS in C. Finally, since C is $\wp-$closed and $\mathfrak{M}^*_\wp$ is $\wp-$complete, by Proposition 3.2, C is $\wp-$complete. Consequently, there exists $w\in C$ such that,

$$\wp_\lambda(u_n,w)\to 0 \text{ as } n\to\infty.$$

Using Inequality (8), we obtain

$$\begin{aligned}
&\wp_\lambda(\varGamma w,w)\le \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(\varGamma w,u_n)+\wp_{\frac{\lambda}{2}}(u_n,w)\right]\\
&\Leftrightarrow \wp_\lambda(\varGamma w,w)\le \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma w,\varGamma u_n)+\mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma u_n,u_n)+\mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n,w)\\
&\Leftrightarrow \wp_\lambda(\varGamma w,w)\le \mathfrak{s}^2k\left[\wp_{\frac{\lambda}{4}}(w,\varGamma w)+\wp_{\frac{\lambda}{4}}(u_n,\varGamma u_n)\right]+\mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma u_n,u_n)+\mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n,w)\\
&\Leftrightarrow \wp_\lambda(\varGamma w,w)\le \mathfrak{s}^2k\wp_{\frac{\lambda}{4}}(w,\varGamma w)+\mathfrak{s}^2ka^n\wp_{\frac{\lambda}{4}}(u_0,\varGamma u_0)+\mathfrak{s}^2a^n\wp_{\frac{\lambda}{4}}(\varGamma u_0,u_0)\\
&\qquad +\mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n,w)\\
&\Leftrightarrow \wp_\lambda(w,\varGamma w)-\mathfrak{s}^2k\wp_{\frac{\lambda}{4}}(w,\varGamma w)\le \mathfrak{s}^2ka^n\wp_{\frac{\lambda}{4}}(u_0,\varGamma u_0)+\mathfrak{s}^2a^n\wp_{\frac{\lambda}{4}}(\varGamma u_0,u_0)+\mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n,w).
\end{aligned}$$

As $n\to\infty$, the right-hand side converges to 0, yielding the following inequality,

$$\wp_\lambda(w,\varGamma w)\le \mathfrak{s}^2k\wp_{\frac{\lambda}{4}}(w,\varGamma w). \tag{12}$$

Iterating this inequality, we get

$$\begin{aligned}
\wp_\lambda(w,\varGamma w)&\le \mathfrak{s}^2k\wp_{\frac{\lambda}{4}}(w,\varGamma w)\\
&\le (\mathfrak{s}^2k)^2\wp_{\frac{\lambda}{4^2}}(w,\varGamma w)\\
&\le (\mathfrak{s}^2k)^3\wp_{\frac{\lambda}{4^3}}(w,\varGamma w)\\
&\ \vdots\\
&\le (\mathfrak{s}^2k)^n\wp_{\frac{\lambda}{4^n}}(w,\varGamma w).
\end{aligned}$$

Using Proposition 3.3 for $\dfrac{\lambda}{4^n}>0$, there exists $K_n>0$ such that,

$$\wp_{\frac{\lambda}{4^n}}(w,\varGamma w)\le K_n.$$

Note that the set $\{K_n | n \in \mathbb{N}\} \subseteq \{K_\lambda | \lambda > 0\}$ and the set $\{K_\lambda | \lambda > 0\}$ is bounded. Hence, there exists $K > 0$ such that,

$$K_n \leq K, \qquad \text{for all } n \in \mathbb{N}.$$

Since $k < \dfrac{1}{\mathfrak{s}^2}$, we have $\mathfrak{s}^2 k < 1$, so $(\mathfrak{s}^2 k) \to 0$ as $n \to \infty$. Therefore,

$$\wp_\lambda(w, \Gamma w) \leq (\mathfrak{s}^2 k)^n \wp_{\frac{\lambda}{4^n}}(w, \Gamma w) \leq (\mathfrak{s}^2 k)^n K \to 0, \qquad \text{as } n \to \infty. \tag{13}$$

Now, let $v_n = \Gamma w$ for all $n \in \mathbb{N}$, we have $\wp_\lambda(v_n, w) \to 0$ as $n \to \infty$. Since $\wp$ satisfies the Δ_2-type condition, which implies that $\wp$ satisfies the Δ_2-condition, we conclude that for all $\lambda > 0$, $\wp_\lambda(v_n, w) \to 0$ as $n \to \infty$. Thus, we obtain

$$\wp_\lambda(w, \Gamma w) = 0 \text{ for all } \lambda > 0, \tag{14}$$

which implies $\Gamma w = w$. Therefore, Γ has at least one FP. □

In [13], a FP theorem for Kannan mapping in MbMS is presented under the condition that the constant $\mathfrak{s} \geq 1$ satisfies the MbMS properties and that k satisfies the conditions for Kannan mappings, with the additional requirement that $\mathfrak{s}k \leq \dfrac{1}{2}$. Furthermore, it is assumed that there exists $u \in \mathfrak{M}^*_\wp$ with $\wp_\lambda(u, \Gamma u) < \infty$. It is important to note that the definitions of convergence and Cauchy sequences in may differ from those in other metric spaces. To address this, we present a similar theorem that applies the Δ_2-type condition, which is particularly suited to the MbMS framework.

Theorem 4.1. *Let $\mathfrak{M}^*_\wp$ be a $\wp-$bounded and $\wp-$complete MbMS such that $\wp$ satisfies the Δ_2-type condition. Suppose $T : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ is a mapping satisfying the condition in (8), with $k \in \left[0, \dfrac{1}{\mathfrak{s}^2}\right)$ and $\lambda > 0$. Further, assume that the set $\{K_\lambda | \lambda > 0\}$, where K_λ satisfies Inequality (6), is bounded. Then, Γ has a unique FP $v \in \mathfrak{M}^*_\wp$ for all $u \in \mathfrak{M}^*_\wp$, the sequence $\{\Gamma^n u\}$ $\wp-$convergent to v. Furthermore, for all $n \in \mathbb{N} \cup \{0\}$, we have*

$$\wp_\lambda(\Gamma^{n+1} u, v) \leq k \left(\frac{k}{1-k}\right)^n \wp_\lambda(u, \Gamma u).$$

Proof. Let u be an arbitrary element of $\mathfrak{M}^*_\wp$. Define $w = \Gamma u$, then we have

$$\wp_\lambda(w, \Gamma w) = \wp_\lambda(\Gamma u, \Gamma w) \leq k \left[\wp_\lambda(u, \Gamma u) + \wp_\lambda(w, \Gamma w)\right].$$

Thus, it follows that,

$$\wp_\lambda(w, \Gamma w) \leq \frac{k}{1-k} \wp_\lambda(u, \Gamma u). \tag{15}$$

Since $k \in \left[0, \dfrac{1}{\mathfrak{s}^2}\right)$, it follows that $\dfrac{k}{1-k} \in [0, 1)$. Next, let $u_0 \in \mathfrak{M}^*_\wp$. Define the sequence $\{u_n\}$ by setting $u_{n+1} = \Gamma u_n$ for all $n \in \mathbb{N} \cup \{0\}$. By Proposition 3.1, since $\mathfrak{M}^*_\wp$ is $\wp-$complete and $\wp$ satisfies the Δ_2-condition, we conclude that $\mathfrak{M}^*_\wp$ is $\wp-$closed. Using Lemma 4.1 and Inequality (15), there exists a point $v \in \mathfrak{M}^*_\wp$ such that the sequence $\{u_n\}$ $\wp-$converges to v and $\Gamma v = v$. Next, suppose there exists $z \in \mathfrak{M}^*_\wp$ such that $\Gamma z = z$. Then,

$$0 \leq \wp_\lambda(z, v) = \wp_\lambda(\Gamma z, \Gamma v) \leq k[\wp_\lambda(z, \Gamma z) + \wp_\lambda(v, \Gamma v)] = 0,$$

which implies $\wp_\lambda(z,v)=0$. Since $\wp$ satisfies the Δ_2-type condition, and following an argument analogous to the proof of (14), we conclude that $\Gamma v = v$. Thus, Γ has a unique FP. Furthermore, for all $u \in \mathfrak{M}^*_\wp$, we have

$$\wp_\lambda(\Gamma^{n+1}u, \Gamma^n u) \leq \frac{k}{1-k}\wp_\lambda(\Gamma^{n-1}u, \Gamma^n u),$$

and consequently,

$$\begin{aligned}
\wp_\lambda(\Gamma^{n+1}u, v) &\leq k\left[\wp_\lambda(\Gamma^n u, \Gamma^{n+1}u) + \wp_\lambda(v, \Gamma v)\right] \\
&\leq k\frac{k}{1-k}\wp_\lambda(\Gamma^{n-1}u, \Gamma^n u) \\
&\leq k\left(\frac{k}{1-k}\right)^n \wp_\lambda(u, \Gamma u), n \in \mathbb{N}\cup\{0\}.
\end{aligned}$$

□

Following [15] and [17], we give the following theorem.

Theorem 4.2. *Let $\mathfrak{M}^*_\wp$ be a MbMS and let $\Gamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ be a mapping such that there exists $\lambda > 0$ for which for all $u, v \in \mathfrak{M}^*_\wp$ with $u \neq v$, the following inequality holds,*

$$\wp_\lambda(\Gamma u, \Gamma v) < \frac{1}{2}\left\{\wp_\lambda(u, \Gamma v) + \wp_\lambda(v, \Gamma v)\right\}. \tag{16}$$

If Γ has a FP, then $\mathfrak{M}^*_\wp$ is $\wp-$complete.

Proof. Suppose $\mathfrak{M}^*_\wp$ is not $\wp-$complete. Then, there exists a $\wp-$CS $\{u_n\} \subset \mathfrak{M}^*_\wp$ that is not $\wp-$convergent. Without loss of generality(briefly w.l.o.g.), assume that each term in $\{u_n\}$ is distinct.

Let $A = \{u_n : n \in \mathbb{N}\}$. Since $\{u_n\}$ is a $\wp-$CS, for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\wp_\lambda(u_n, u_m) < \varepsilon$ for all $n, m \in \mathbb{N}$ with $n, m \geq n_\varepsilon$. However, since $\{u_n\}$ does not $\wp-$converge in $\mathfrak{M}^*_\wp$, we have $\wp_\lambda(u, A) > 0$ for all $u \in \mathfrak{M}^*_\wp - A$. Next, let $x \in \mathfrak{M}^*_\wp$. We examine the following two cases:

- If $u \in \mathfrak{M}^*_\wp - A$, then $\frac{1}{2}\wp_\lambda(u, A) > 0$. Since $\{u_n\}$ is a $\wp-$CS, then there exists $n_u \in \mathbb{N}$ such that for all $m \in \mathbb{N}$ with $m \geq n_u$,

$$\wp_\lambda(u_m, u_{n_u}) < \frac{1}{2}\wp_\lambda(u, A).$$

  Therefore, we obtain

$$\wp_\lambda(u_m, u_{n_u}) < \frac{1}{2}\wp_\lambda(u, u_n). \tag{17}$$

- If $u \in A$, then $u = u_{n_0}$ for some $n_0 \in \mathbb{N}$. Since all terms of the sequence $\{u_n\}$ are distinct, we have $\dfrac{1}{2}\wp_\lambda(u_n, u_{n_0}) > 0$ for $n \neq n_0$. Furthermore, since $\{u_n\}$ is a $\wp-$CS, there exists $n_0' \in \mathbb{N}$ with $n_0' > n_0$ such that for all $m \in \mathbb{N}$ with $m \geq n_0'$,

$$\wp_\lambda(u_m, u_{n_0'}) < \frac{1}{2}\wp_\lambda(u_n, u_{n_0}). \tag{18}$$

Based on Inequalities (17) and (18), we define a mapping $\Gamma : \mathfrak{M}_\wp^* \to \mathfrak{M}_\wp^*$ by,

$$\Gamma u = \begin{cases} u_{n_u}, & \text{if} \quad u \in \mathfrak{M}_\wp^* - A, \\ u_{n_0'}, & \text{if} \quad u \in A \text{ and } u = u_{n_0}. \end{cases}$$

First, we show that Γ does not have a FP, i.e. for all $u \in \mathfrak{M}_\wp^*$, $\wp_\lambda(u, \Gamma u) > 0$. Let $u \in \mathfrak{M}_\wp^*$. Then, we consider two cases, specifically $u \in \mathfrak{M}_\wp^* - A$ or $u \in A$:

- If $u \in \mathfrak{M}_\wp^* - A$, then $\Gamma u = u_{n_u}$. From Inequality (17), we obtain that for all $m \in \mathbb{N}$ with $m \geq n_u$,

$$\wp_\lambda(u_m, u_{n_u}) < \frac{1}{2}\wp_\lambda(u, u_{n_u}) = \frac{1}{2}\wp_\lambda(u, \Gamma u).$$

  Therefore,

$$\wp_\lambda(u, \Gamma u) > 2\wp_\lambda(u_m, u_{n_u}) \geq 0.$$

- If $u \in A$, then $u = u_{n_0}$ for some $n_0 \in \mathbb{N}$, and $\Gamma u = u_{n_0'}$. By applying Inequality (18), we obtain

$$\wp_\lambda(u_m, u_{n_0'}) < \frac{1}{2}\wp_\lambda(u_{n_0'}, u_{n_0}) = \frac{1}{2}\wp_\lambda(\Gamma u, u).$$

  Thus, it follows that,

$$\wp_\lambda(u, \Gamma u) > 2\wp_\lambda(u_m, u_{n_0'}) \geq 0.$$

Thus, we conclude that Γ does not have a FP. Next, we will show that Γ satisfies Inequality (16). Let $u, v \in \mathfrak{M}_\wp^*$ with $u \neq v$,

- If $u, v \in \mathfrak{M}_\wp^* - A$, then $\Gamma u = u_{n_u}$ and $\Gamma v = v_{n_v}$. W.l.o.g., assume that $n_v \geq n_u$. From Inequality (17), we obtain

$$\wp_\lambda(\Gamma u, \Gamma v) = \wp_\lambda(u_{n_u}, v_{n_v}) < \frac{1}{2}\wp_\lambda(u, u_{n_u}) = \frac{1}{2}\wp_\lambda(u, \Gamma u). \tag{19}$$

- If $u, v \in A$, then $u = u_{n_0}$ and $v = v_{m_0}$ for some $n_0, m_0 \in \mathbb{N}$. Therefore, we have $\Gamma u = u_{n_0'}$ and $\Gamma v = v_{m_0'}$. W.l.o.g., assume that $m_0' \geq n_0'$. By using Inequality (18), we get

$$\wp_\lambda(\Gamma u, \Gamma v) = \wp_\lambda(u_{n_0'}, v_{m_0'}) < \frac{1}{2}\wp_\lambda(u_{n_0'}, u_{n_0}) = \frac{1}{2}\wp_\lambda(u, \Gamma u). \tag{20}$$

- If $u \in \mathfrak{M}_\wp^* - A$ and $v \in A$, then $v = u_{n_0}$ for some $n_0 \in \mathbb{N}$, such that $\Gamma u = u_{n_u}$ and $\Gamma v = u_{n_0'}$. If $n_0' \geq n_u$, then by using Inequality (17), we obtain

$$\wp_\lambda(\Gamma u, \Gamma v) = \wp_\lambda(u_{n_0'}, u_{n_u}) < \frac{1}{2}\wp_\lambda(u, u_{n_u}) = \frac{1}{2}\wp_\lambda(u, \Gamma u). \tag{21}$$

  If $n_u > n_0'$, then by applying Inequality (18), we get

$$\wp_\lambda(u_{n_u}, u_{n_0'}) < \frac{1}{2}\wp_\lambda(u_{n_0'}, u_{n_0}) = \frac{1}{2}\wp_\lambda(v, \Gamma v). \tag{22}$$

Thus, from Inequalities (19), (20), (21), (22), it follows that Γ satisfies Inequality (16). So, there is a contradiction. This leads to a contradiction, which implies that $\mathfrak{M}_\wp^*$ must be $\wp-$complete. □

4.2 Some theorems using subadditive altering distance function

Now, we give a theorem based on [16], [17], and [29].

Theorem 4.3. *Let $\mathfrak{M}^*_\wp$ be a $\wp-$bounded and $\wp-$complete MbMS such that $\wp$ satisfies the Δ_2-type condition. Suppose that $\varGamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ is a mapping such that there exists $\lambda > 0$, a subadditive altering distance function $\mho$, and nonnegative numbers p_1, p_2, p_3 with $p_1 + \mathfrak{s}(p_2 + p_3) < 1$, such that for all $u, v \in \mathfrak{M}^*_\wp$,*

$$\mho(\wp_\lambda(\varGamma u, \varGamma v)) \leq p_1\mho(\wp_\lambda(u, \varGamma u)) + p_2\mho(\wp_\lambda(v, \varGamma v)) + p_3\mho(\wp_\lambda(u, v)). \tag{23}$$

Further, assume that the set $\{K_\lambda|\lambda > 0\}$, where K_λ satisfies Inequality (6), is bounded. Then, $\varGamma$ has a unique FP $z \in \mathfrak{M}^*_\wp$ such that the sequence $\{\varGamma^n u\}$ $\wp-$converges to z, and there exists $p = \dfrac{p_2 + p_3}{1 - p_1} \in [0, 1)$ such that,

$$\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) \leq p^n \wp_\lambda(u, \varGamma u), n \in \mathbb{N} \cup \{0\}.$$

Proof. Let $u \in \mathfrak{M}^*_\wp$ be an arbitrary element and define the iteration sequence $\{\varGamma^n u\}$. Based on Inequality (23), we obtain for all $n \in \mathbb{N}$,

$$\begin{aligned} \mho(\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) &\leq p_1\mho(\wp_\lambda(\varGamma^n u, \varGamma^{n+1}u) \\ &\quad + p_2\mho(\wp_\lambda(\varGamma^{n-1}u, \varGamma^n u) + p_3\mho(\wp_\lambda(\varGamma^n u, \varGamma^{n-1}u) \\ \Leftrightarrow (1 - p_1)\mho(\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) &\leq (p_2 + p_3)\mho(\wp_\lambda(\varGamma^{n-1}u, \varGamma^n u). \end{aligned}$$

Thus, it follows that,

$$\mho\left(\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u)\right) \leq \frac{p_2 + p_3}{1 - p_1}\mho\left(\wp_\lambda(\varGamma^{n-1}u, \varGamma^n u)\right). \tag{24}$$

By taking $q = \dfrac{p_2 + p_3}{1 - p_1}$, we know that $p \in [0, 1)$ and for all $n \in \mathbb{N}$, we obtain

$$\mho(\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u)) \leq q\, \mho\left(\wp_\lambda(\varGamma^{n-1}u, \varGamma^n u)\right). \tag{25}$$

Now, applying Proposition 2.2, there exists $q' \in [0, 1)$, such that $n \in \mathbb{N}$,

$$\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) \leq q' \wp_\lambda(\varGamma^{n-1}u, \varGamma^n u). \tag{26}$$

Without loss of generality, assume that $q = q'$. Therefore, for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) \leq q^n \wp_\lambda(u, \varGamma u). \tag{27}$$

Since $p \in [0, 1)$, it follows that $q^n \to 0$ as $n \to \infty$, implying,

$$\wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) \to 0, \qquad \text{as } n \to \infty.$$

Next, let $m, n \in \mathbb{N}$ such that $m > n$. If $m = n + 1$, then we have

$$\wp_\lambda(\varGamma^m u, \varGamma^n u) = \wp_\lambda(\varGamma^{n+1}u, \varGamma^n u) \leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u).$$

If $m = n + 2$, then,

$$\begin{aligned} \wp_\lambda(\varGamma^m u, \varGamma^n u) &= \wp_\lambda(\varGamma^{n+2}u, \varGamma^n u) \\ &\leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+2}u, \varGamma^{n+1}u) + \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u) \\ &\leq \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma^{n+2}u, \varGamma^{n+1}u) + \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u). \end{aligned}$$

If $m = n+3$, then,

$$\begin{aligned}\wp_\lambda(\varGamma^m u, \varGamma^n u) &= \wp_\lambda(\varGamma^{n+3}u, \varGamma^n u)\\ &\leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+3}u, \varGamma^{n+1}u) + \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u)\\ &\leq \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma^{n+3}u, \varGamma^{n+2}u) + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma^{n+2}u, \varGamma^{n+1}u) + \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u)\\ &\leq \mathfrak{s}^3\wp_{\frac{\lambda}{8}}(\varGamma^{n+3}u, \varGamma^{n+2}u) + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma^{n+2}u, \varGamma^{n+1}u) + \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u).\end{aligned}$$

If the process continues for all $m, n \in \mathbb{N}$ such that $m > n$, we obtain

$$\wp_\lambda(\varGamma^m u, \varGamma^n u) \leq \mathfrak{s}^{m-n}\wp_{\frac{\lambda}{2^{m-n}}}(\varGamma^m u, \varGamma^{m-1}u) + \cdots + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\varGamma^{n+2}u, \varGamma^{n+1}u) + \mathfrak{s}\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma^n u).$$

Thus, based on Inequality (27), it follows that,

$$\wp_\lambda(\varGamma^m u, \varGamma^n u) \leq \mathfrak{s}^{m-n}q^{m-1}\wp_{\frac{\lambda}{2^{m-n}}}(u, \varGamma u) + \cdots + \mathfrak{s}^2 q^{n+1}\wp_{\frac{\lambda}{4}}(u, \varGamma u) + \mathfrak{s}q^n\wp_{\frac{\lambda}{2}}(u, \varGamma u).$$

Furthermore, since $\dfrac{\lambda}{2^m} < \dfrac{\lambda}{2^{m-n}} < \cdots < \dfrac{\lambda}{4} < \dfrac{\lambda}{2}$, using (3), we get

$$\begin{aligned}\wp_\lambda(\varGamma^m u, \varGamma^n u) &\leq \mathfrak{s}^{m-n+1}q^{m-1}\wp_{\frac{\lambda}{2^m}}(u, \varGamma u) + \cdots + \mathfrak{s}^3 q^{n+1}\wp_{\frac{\lambda}{2^m}}(u, \varGamma u) + \mathfrak{s}^2 q^n\wp_{\frac{\lambda}{2^m}}(u, \varGamma u)\\ &\leq \mathfrak{s}q^{n-1}\wp_{\frac{\lambda}{2^m}}(u, \varGamma u)\left(\mathfrak{s}q + (\mathfrak{s}q)^2 + \cdots + (\mathfrak{s}q)^m\right)\\ &\leq \mathfrak{s}q^{n-1}\wp_{\frac{\lambda}{2^m}}(u, \varGamma u)\left(\mathfrak{s}q + (\mathfrak{s}q)^2 + \cdots\right).\end{aligned}$$

Note that $0 \leq \mathfrak{s}q < 1$, so it follows that,

$$\wp_\lambda(\varGamma^m u, \varGamma^n u) \leq \frac{\mathfrak{s}^2 q^n}{1-\mathfrak{s}q}\wp_{\frac{\lambda}{2^m}}(u, \varGamma u).$$

By an argument analogous to the proof of (13), there exists $K > 0$ such that,

$$\wp_{\frac{\lambda}{2^m}}(u, \varGamma u) \leq \frac{\mathfrak{s}^2 q^n K}{1-\mathfrak{s}q}, \qquad \text{for all } n \in \mathbb{N}.$$

As a result, we conclude that,

$$\wp_\lambda(\varGamma^m u, \varGamma^n u) \to 0, \qquad \text{as } m, n \to \infty.$$

Thus, the sequence $\{\varGamma^n u\}$ is a $\wp-$CS in $\mathfrak{M}^*_\wp$. Since $\mathfrak{M}^*_\wp$ is $\wp-$complete, there exists $z \in \mathfrak{M}^*_\wp$ such that $\wp_\lambda(\varGamma^n u, z) \to 0$ as $n \to \infty$. To show that z is a FP, we need to show $\mho(\wp_\lambda(\varGamma^{n+1}u, \varGamma z)) \to 0$ as $n \to \infty$. Based on Inequality (23), we obtain

$$\begin{aligned}\mho(\wp_\lambda(\varGamma^{n+1}u, \varGamma z)) &\leq p_1\mho(\wp_\lambda(\varGamma^n u, \varGamma^{n+1}u)) + p_2\mho\left(\wp_\lambda(z, \varGamma z)\right) + p_3\mho(\wp_\lambda(\varGamma^n u, z))\\ &\leq p_1\mho(\wp_\lambda(\varGamma^n u, \varGamma^{n+1}u)) + p_2\mho\left(\mathfrak{s}\left[\wp_{\frac{\lambda}{2}}z, \varGamma^{n+1}u) + \wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma z)\right]\right)\\ &\quad + p_3\mho(\wp_\lambda(\varGamma^n u, z))\\ &\leq p_1\mho(\wp_\lambda(\varGamma^n u, \varGamma^{n+1}u)) + \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(z, \varGamma^{n+1}u)\right) + \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma z)\right)\\ &\quad + p_3\mho(\wp_\lambda(\varGamma^n u, z)).\end{aligned}$$

Thus, we have

$$\begin{aligned}\mho(\wp_\lambda(\varGamma^{n+1}u, \varGamma z)) - \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(\varGamma^{n+1}u, \varGamma z)\right) &\leq p_1\mho(\wp_\lambda(\varGamma^n u, \varGamma^{n+1}u))\\ &\quad + \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(z, \varGamma^{n+1}u)\right) + p_3\mho(\wp_\lambda(\varGamma^n u, z)).\end{aligned} \tag{28}$$

Since $\mho$ is an altering distance function and $\wp_\lambda(\Gamma^n u, z) \to 0$ as $n \to \infty$, it follows that,

$$\mho(\wp_\lambda(\Gamma^n u, z)) \to 0, \qquad \text{as } n \to \infty.$$

Also, since $\wp$ satisfies the Δ_2-condition and $\wp_\lambda(\Gamma^n u, z) \to 0$ as $n \to \infty$, we conclude that,

$$\wp_{\frac{\lambda}{2}}(\Gamma^n u, z) \to 0, \qquad \text{as } n \to \infty.$$

Therefore, we have

$$\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma^n u, z)\right) \to 0, \qquad \text{as } n \to \infty.$$

From Inequality (28), we get

$$\begin{aligned}\mho(\wp_\lambda(\Gamma^{n+1}u, \Gamma z)) - \mathfrak{s}p_2\mho(\wp_{\frac{\lambda}{2}}(\Gamma^{n+1}u, \Gamma z)) &\leq p_1\mho(\wp_\lambda(\Gamma^n u, \Gamma^{n+1}u)) + \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(z, \Gamma^{n+1}u)\right)\\ &\quad + p_3\mho(\wp_\lambda(\Gamma^n u, z)) \to 0, \qquad \text{as } n \to \infty.\end{aligned}$$

Hence, it follows that,

$$\mho(\wp_\lambda(\Gamma^n u, \Gamma z)) \leq \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma^{n+1}u, \Gamma z)\right).$$

Using this, we obtain

$$\begin{aligned}\mho\left(\wp_\lambda(\Gamma^n u, \Gamma z)\right) &\leq \mathfrak{s}p_2\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma^{n+1}u, \Gamma z)\right)\\ &\leq (\mathfrak{s}p_2)^2\mho\left(\wp_{\frac{\lambda}{2^2}}(\Gamma^{n+1}u, \Gamma z)\right)\\ &\leq (\mathfrak{s}p_2)^3\mho\left(\wp_{\frac{\lambda}{2^3}}(\Gamma^{n+1}u, \Gamma z)\right)\\ &\ \ \vdots\\ &\leq (\mathfrak{s}p_2)^n\mho\left(\wp_{\frac{\lambda}{2^n}}(\Gamma^{n+1}u, \Gamma z)\right).\end{aligned}$$

Again, by an argument analogous to the proof of (13), there exists $K > 0$ such that,

$$\mho\left(\wp_\lambda\left(\Gamma^n u, \Gamma z\right)\right) \leq (\mathfrak{s}p_2)^n\mho\left(\wp_{\frac{\lambda}{2^n}}\left(\Gamma^{n+1}u, \Gamma z\right)\right) \leq (\mathfrak{s}p_2)^n K, \qquad \text{for all } n \in \mathbb{N}.$$

Since $\mathfrak{s}p_2 \in [0, 1)$, it follows that,

$$\mho\left(\wp_\lambda(\Gamma^n u, \Gamma z)\right) \leq (\mathfrak{s}p_2)^n K \to 0, \qquad \text{as } n \to \infty. \tag{29}$$

Furthermore, since $\wp$ satisfies the Δ_2-condition, we conclude

$$\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma^n u, \Gamma z)\right) \to 0, \qquad \text{as } n \to \infty.$$

As a result, we have

$$\mho\left(\wp_\lambda(z, \Gamma z)\right) \leq \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma^{n+1}u, z)\right) + \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma^{n+1}u, \Gamma z)\right) \to 0, \qquad \text{as } n \to \infty.$$

Thus, $\mho(\wp_\lambda(z, \Gamma z)) = 0$. So, it follows that $\wp_\lambda(z, \Gamma z) = 0$. Since $\wp$ satisfies the Δ_2-condition, then $\Gamma z = z$. Next, suppose $w, z \in \mathfrak{M}^*_\wp$ are two different fixed points of Γ. Note that,

$$\mho(\wp_\lambda(w, z)) \leq p_1\mho(\wp_\lambda(w, \Gamma w)) + p_2\mho(\wp_\lambda(z, \Gamma z)) + p_3\mho(\wp_\lambda(w, z)) \leq p_3\mho(\wp_\lambda(w, z)),$$

so $p_3 \geq 1$. This leads to a contradiction. Therefore, it is true that z is a unique FP of Γ. □

Furthermore, we also give the following theorem using subadditive altering distance function.

Theorem 4.4. *Let $(\mathfrak{M}, \wp)$ be a MbMS and $\Gamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ be a mapping such that there exists $\lambda > 0$, a subadditive altering distance function $\mho$, and $p \in \left(0, \dfrac{1}{\mathfrak{s}+1}\right)$ satisfying*

$$\mho(\wp_\lambda(\Gamma u, \Gamma v)) \leq p\left\{\mho(\wp_\lambda(u, \Gamma u)) + \mho(\wp_\lambda(v, \Gamma v)) + \mho(\wp_\lambda(u, v))\right\}, \tag{30}$$

for all $u, v \in \mathfrak{M}^*_\wp$. If Γ has a FP, then $\mathfrak{M}^*_\wp$ is $\wp-$complete.

Proof. The proof is analogous to Theorem 4.2. □

Now, we give the following theorem.

Theorem 4.5. *Let $\mathfrak{M}^*_\wp$ be a $\wp-$bounded MbMS such that $\wp$ satisfies the Δ_2-type condition. Assume that the set $\{K_\lambda | \lambda > 0\}$, where K_λ satisfies Inequality (6), is bounded. If $\Gamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ is a mapping such that there exists $\lambda > 0$ and $p \in \left[0, \dfrac{1}{s}\right)$, subadditive altering distance function $\mho$ such that,*

$$\mho(\wp_\lambda(\Gamma u, \Gamma v)) \leq p\{\mho(\wp_\lambda(u, \Gamma u)) + \mho(\wp_\lambda(v, \Gamma v))\}, \tag{31}$$

for all $u, v \in \mathfrak{M}^*_\wp$, then Γ has a unique FP $z \in \mathfrak{M}^*_\wp$, for all $u \in \mathfrak{M}^*_\wp$, $\{\Gamma^n u\}$ $\wp-$convergent to $z \in \mathfrak{M}^*_\wp$ and for some $q = \dfrac{p}{1-p} \in [0, 1)$,

$$\wp_\lambda(\Gamma^{n+1} u, z) \leq q^n \wp_\lambda(u, \Gamma u), \qquad n \in \mathbb{N} \cup \{0\}.$$

Proof. Let u be an arbitrary element in $\mathfrak{M}^*_\wp$. By taking $w = \Gamma u$ and using Inequality (31), we obtain

$$\begin{aligned}
\mho(\wp_\lambda(w, \Gamma w)) &= \mho(\wp_\lambda(\Gamma u, \Gamma w)) \\
\Leftrightarrow \mho(\wp_\lambda(w, \Gamma w)) &\leq p\{\mho(\wp_\lambda(u, \Gamma u)) + \mho(\wp_\lambda(w, \Gamma w))\} \\
\Leftrightarrow (1-p)\mho(\wp_\lambda(w, \Gamma w)) &\leq p\mho(\wp_\lambda(u, \Gamma u)) \\
\Leftrightarrow \mho(\wp_\lambda(w, \Gamma w)) &\leq \frac{p}{1-p}\mho(\wp_\lambda(w, \Gamma w)).
\end{aligned}$$

Since $p \in \left[0, \dfrac{1}{\mathfrak{s}}\right)$, we define $q = \dfrac{p}{1-p}$, which implies $q \in [0, 1)$. Consequently,

$$\mho(\wp_\lambda(w, \Gamma w)) \leq q\, \mho(\wp_\lambda(w, \Gamma w)). \tag{32}$$

By applying Proposition 2.2 and noting that Inequality (32) satisfies Inequality (4), it follows that,

$$\wp_\lambda(w, \Gamma w) \leq q'\, \wp_\lambda(w, \Gamma w), \qquad \text{for some } q' \in (0, 1). \tag{33}$$

W.l.o.g., we can assume that $q' = q$.

Next, let be $u_0 \in \mathfrak{M}^*_\wp$ an arbitrary element. Consider the sequence $\{u_n\}$ defined by,

$$u_{n+1} = \Gamma u_n, \qquad n \in \mathbb{N} \cup \{0\}.$$

We will show that $\{u_n\}$ is a $\wp-$CS. Let $m, n \in \mathbb{N} \cup \{0\}$ such that $m > n$. If $m = n+1$, then,

$$\wp_\lambda(u_n, u_m) = \wp_\lambda(u_n, u_{n+1}) \leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}).$$

If $m = n + 2$, then,

$$\begin{aligned}\wp_\lambda(u_n, u_m) &= \wp_\lambda(u_n, u_{n+2})\\ &\leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \wp_{\frac{\lambda}{2}}(u_{n+1}, u_{n+2})\right]\\ &\leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \mathfrak{s}\left[\wp_{\frac{\lambda}{4}}(u_{n+1}, u_{n+1}) + \wp_{\frac{\lambda}{4}}(u_{n+1}, u_{n+2})\right]\right]\\ &= \mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \mathfrak{s}^2\wp_{\frac{\lambda}{2^2}}(u_{n+1}, u_{n+2}).\end{aligned}$$

If $m = n + 3$, then,

$$\begin{aligned}\wp_\lambda(u_n, u_m) &= \wp_\lambda(u_n, u_{n+3})\\ &\leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \wp_{\frac{\lambda}{2}}(u_{n+1}, u_{n+3})\right]\\ &\leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \mathfrak{s}\left[\wp_{\frac{\lambda}{4}}(u_{n+1}, u_{n+2}) + \wp_{\frac{\lambda}{4}}(u_{n+2}, u_{n+3})\right]\right]\\ &\leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \mathfrak{s}^2\wp_{\frac{\lambda}{2^2}}(u_{n+1}, u_{n+2}) + \mathfrak{s}^3\wp_{\frac{\lambda}{2^3}}(u_{n+2}, u_{n+3}).\end{aligned}$$

Continuing this process for all $m, n \in \mathbb{N} \cup \{0\}$ such that $m > n$, we have

$$\wp_\lambda(u_n, u_m) \leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \mathfrak{s}^2\wp_{\frac{\lambda}{2^2}}(u_{n+1}, u_{n+2}) + \cdots + \mathfrak{s}^{m-n}\wp_{\frac{\lambda}{2^{m-n}}}(u_{m-1}, u_m). \tag{34}$$

Using Inequality (31), we have

$$\begin{aligned}\mho(\wp_\lambda(u_1, u_2)) &= \mho(\wp_\lambda(\varGamma u_0, \varGamma u_1))\\ &\leq p\left[\mho(\wp_\lambda(u_0, \varGamma u_0)) + \mho(\wp_\lambda(u_1, \varGamma u_1))\right].\end{aligned}$$

By applying the same approach as in Inequality (33), it follows that,

$$\wp_\lambda(u_1, u_2) \leq q\wp_\lambda(u_0, \varGamma u_0).$$

Similarly, we also have

$$\wp_\lambda(u_2, u_3) \leq q\wp_\lambda(u_1, u_2),$$

and by iteration, we obtain

$$\wp_\lambda(u_2, u_3) \leq q\wp_\lambda(u_1, u_2) \leq q^2\wp_\lambda(u_0, \varGamma u_0).$$

Continuing this process for all $n \in \mathbb{N}$, we have

$$\wp_\lambda(u_n, u_{n+1}) \leq q^n\wp_\lambda(u_0, \varGamma u_0).$$

Now, using Inequality (34) for all $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\wp_\lambda(u_n, u_m) &\leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(u_n, u_{n+1}) + \mathfrak{s}^2\wp_{\frac{\lambda}{2^2}}(u_{n+1}, u_{n+2}) + \cdots + \mathfrak{s}^{m-n}\wp_{\frac{\lambda}{2^{m-n}}}(u_{m-1}, u_m)\\ &\leq \mathfrak{s}q^n\wp_{\frac{\lambda}{2}}(u_0, \varGamma u_0) + \mathfrak{s}^2q^{n+1}\wp_{\frac{\lambda}{2^2}}(u_0, \varGamma u_0) + \cdots + \mathfrak{s}^{m-n}q^{m-1}\wp_{\frac{\lambda}{2^{m-n}}}(u_0, \varGamma u_0).\end{aligned}$$

Furthermore, since $\frac{\lambda}{2^m} < \frac{\lambda}{2^{m-n}} < \cdots < \frac{\lambda}{2^2} < \frac{\lambda}{2}$ for all $m, n \in \mathbb{N}$ and using Inequality (3), it follows that,

$$\begin{aligned}\wp_\lambda(u_n, u_m) &\leq \mathfrak{s}q^n\wp_{\frac{\lambda}{2}}(u_0, \varGamma u_0) + \mathfrak{s}^2q^{n+1}\wp_{\frac{\lambda}{2^2}}(u_0, \varGamma u_0) + \cdots + \mathfrak{s}^{m-n}q^{m-1}\wp_{\frac{\lambda}{2^{m-n}}}(u_0, \varGamma u_0)\\ &\leq \mathfrak{s}^2q^n\wp_{\frac{\lambda}{2^m}}(u_0, \varGamma u_0) + \mathfrak{s}^3q^{n+1}\wp_{\frac{\lambda}{2^m}}(u_0, \varGamma u_0) + \cdots + \mathfrak{s}^{m-n+1}q^{m-1}\wp_{\frac{\lambda}{2^m}}(u_0, \varGamma u_0)\\ &\leq q^{n-2}\wp_{\frac{\lambda}{2^m}}(u_0, \varGamma u_0)((\mathfrak{s}q)^2 + (\mathfrak{s}q)^3 + \cdots + (\mathfrak{s}q)^{m-n+1})\\ &\leq q^{n-2}\wp_{\frac{\lambda}{2^m}}(u_0, \varGamma u_0)((\mathfrak{s}q)^2 + (\mathfrak{s}q)^3 + \cdots).\end{aligned}$$

Since $p \in \left[0, \dfrac{1}{2\mathfrak{s}}\right)$, it follows that $\mathfrak{s}q \in [0,1)$. Therefore, for all $m, n \in \mathbb{N}$,

$$\wp_\lambda(u_n, u_m) \leq \frac{\mathfrak{s}^2 q^n}{1-\mathfrak{s}q}\wp_{\frac{\lambda}{2^m}}(u_0, \Gamma u_0).$$

By an argument analogous to the proof of (13), there exists $K > 0$ such that,

$$\wp_\lambda(u_n, u_m) \leq \frac{\mathfrak{s}^2 q^n K}{1-\mathfrak{s}q}, \qquad \text{for all } m, n \in \mathbb{N}.$$

Finally, since $q \in [0,1)$, it follows that,

$$\wp_\lambda(u_n, u_m) \to 0, \qquad \text{as } m, n \to \infty.$$

Hence, the sequence $\{u_n\}$ is a $\wp-$CS. Using the completeness of $\mathfrak{M}^*_\wp$, there exists $z \in \mathfrak{M}^*_\wp$, such that $\wp_\lambda(u_n, z) \to 0$ as $n \to \infty$. Consequently, we have

$$\mho(\wp_\lambda(u_n, z)) \to 0, \qquad \text{as } n \to \infty.$$

It is easy to show that for all $n \in \mathbb{N}$,

$$\wp_\lambda(\Gamma z, z) \leq \mathfrak{s}\wp_{\frac{\lambda}{2}}(\Gamma z, \Gamma u_n) + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\Gamma u_n, u_n) + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(u_n, z).$$

without loss of generality, assume that $\mathfrak{s}$ a positive integer. Using the properties of subadditive altering distance functions, we get

$$\begin{aligned}\mho(\wp_\lambda(\Gamma z, z)) &\leq \mho\left(\mathfrak{s}\wp_{\frac{\lambda}{2}}(\Gamma z, \Gamma u_n) + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(\Gamma u_n, u_n) + \mathfrak{s}^2\wp_{\frac{\lambda}{4}}(u_n, z)\right) \\ &\leq \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma z, \Gamma u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(\Gamma u_n, u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right).\end{aligned}$$

Then, using Inequality (31), we have

$$\begin{aligned}\mho(\wp_\lambda(\Gamma z, z)) &\leq \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(\Gamma z, \Gamma u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(\Gamma u_n, u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right) \\ &\leq \mathfrak{s}p\left\{\mho\left(\wp_{\frac{\lambda}{2}}(z, \Gamma z)\right) + \mho\left(\wp_{\frac{\lambda}{2}}(u_n, \Gamma u_n)\right)\right\} + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(\Gamma u_n, u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right) \\ &\leq \mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}(z, \Gamma z)\right) + \mathfrak{s}p\mho\left(\mathfrak{s}\wp_{\frac{\lambda}{4}}(u_n, \Gamma u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(\Gamma u_n, u_n)\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right) \\ &\leq \mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}(z, \Gamma z)\right) + \mathfrak{s}^2p\mho\left(\wp_{\frac{\lambda}{4}}(u_n, u_{n+1})\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, u_{n+1})\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right).\end{aligned} \tag{35}$$

Since $\wp$ satisfies the Δ_2-condition and $\wp_\lambda(u_n, z) \to 0$, $\wp_\lambda(u_m, u_n) \to 0$ as $m, n \to \infty$, we have

$$\wp_{\frac{\lambda}{4}}(u_n, z) \to 0 \qquad \text{and} \qquad \wp_{\frac{\lambda}{4}}(u_m, u_n) \to 0, \qquad \text{as } m, n \to \infty.$$

Consequently,

$$\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right) \to 0 \qquad \text{and} \qquad \mho\left(\wp_{\frac{\lambda}{4}}(u_m, u_n)\right) \to 0, \qquad \text{as } m, n \to \infty.$$

From (35), it follows that for $n \to \infty$,

$$\begin{aligned}\mho(\wp_\lambda(\Gamma z, z)) - \mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}(z, \Gamma z)\right) &\leq \mathfrak{s}^2p\mho\left(\wp_{\frac{\lambda}{4}}(u_n, u_{n+1})\right) \\ &\quad + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, u_{n+1})\right) + \mathfrak{s}^2\mho\left(\wp_{\frac{\lambda}{4}}(u_n, z)\right) \to 0.\end{aligned}$$

Thus,

$$\mho(\wp_\lambda(\Gamma z, z)) \le \mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}(z, \Gamma z)\right).$$

Using an argument analogous to the proof of (29), there exists $K > 0$ such that,

$$\mho(\wp_\lambda(\Gamma z, z)) \le (\mathfrak{s}p)^n K, \qquad \text{for all } n \in \mathbb{N}.$$

Since $\mathfrak{s}p \in [0, 1)$, we have

$$\mho(\wp_\lambda(\Gamma z, z)) \to 0, \qquad \text{as } n \to \infty.$$

This implies $\mho(\wp_\lambda(\Gamma z, z)) = 0$, and therefore $\wp_\lambda(\Gamma z, z) = 0$. By Δ_2-condition, $\Gamma z = z$. So, z is a FP of Γ. Now, we show the uniqueness of the FP. Suppose $w \in \mathfrak{M}^*_\wp$ such that $\Gamma w = w$. Using Inequality (31),

$$\mho(\wp_\lambda(w, z)) = \mho(\wp_\lambda(\Gamma w, \Gamma z)) \le p\left\{\mho(\wp_\lambda(w, \Gamma w)) + \mho\left(\wp_\lambda(z, \Gamma z)\right)\right\} = 0.$$

Thus, $\wp_\lambda(w, z) = 0$. By the Δ_2-condition, then $w = z$. Hence, z is a unique FP of Γ. Furthermore, a sequence $\{\Gamma^n u\}$ is defined. Using Inequality (31), we get

$$\mho\left(\wp_\lambda\left(\Gamma^{n+1}u, \Gamma^n u\right)\right) \le q\,\mho\left(\wp_\lambda\left(\Gamma^{n-1}u, \Gamma^n u\right)\right), \qquad \text{for all } n \in \mathbb{N}.$$

Based on Proposition 2.2, it follows that,

$$\wp_\lambda\left(\Gamma^{n+1}u, \Gamma^n u\right) \le q\,\wp_\lambda\left(\Gamma^{n-1}u, \Gamma^n u\right), \qquad \text{for all } n \in \mathbb{N}.$$

Thus, we have

$$\wp_\lambda\left(\Gamma^{n+1}u, \Gamma^n u\right) \le q^n\,\wp_\lambda\left(u, \Gamma u\right), \qquad \text{for all } n \in \mathbb{N}.$$

Since $q \in [0, 1)$, we have $\wp_\lambda\left(\Gamma^{n+1}u, \Gamma^n u\right) \to 0$, as $n \to \infty$. Using Inequality (31) for all $n \in \mathbb{N}$,

$$\mho\left(\wp_\lambda\left(\Gamma^{n+1}u, z\right)\right) \le p\,\mho\left(\wp_\lambda\left(\Gamma^n u, \Gamma^{n+1}u\right)\right).$$

As a result, we have

$$\mho\left(\wp_\lambda\left(\Gamma^{n+1}u, z\right)\right) \to 0, \qquad \text{as } n \to \infty.$$

Thus, $\wp_\lambda\left(\Gamma^{n+1}u, z\right) \to 0$ as $n \to \infty$. In other words, $\{\Gamma^n u\}$ $\wp-$convergent to z. □

Furthermore, we also give the following theorem.

Theorem 4.6. *Let $\mathfrak{M}^*_\wp$ be a MbMS such that $\wp$ satisfies the Δ_2-condition. If $\Gamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ satisfies the condition given in (31) with $p \in \left(0, \frac{1}{\mathfrak{s}}\right)$, then $\mathfrak{M}^*_\wp$ is $\wp-$complete.*

Proof. The proof follows analogousy to that of Theorem 4.2. □

The following theorem is based on the results from [17] where the $\mathfrak{M}^*_\wp$ is assumed $\wp-$boundedly compact. In this theorem, the parameter p, which satisfies Inequality (31), is independent of the coefficient $\mathfrak{s}$ that satisfies Axiom $(B3)$ in Definition 2.3. The MbM $\wp$ is sufficient to satisfy the Δ_2-condition, it does not need to satisfy the Δ_2-type condition. Thus, the boundedness of the set $\{K_\lambda | \lambda > 0\}$ in (7) is not required.

Theorem 4.7. *Let $\mathfrak{M}^*_\wp$ be a $\wp-$boundedly compact MbMS where $\wp$ satisfies the Δ_2-condition. If $\Gamma : \mathfrak{M}^*_\wp \to \mathfrak{M}^*_\wp$ is a $\wp-$continuous mapping satisfying (31) with $p \in \left[0, \frac{1}{2}\right)$, then Γ has a unique FP $w \in \mathfrak{M}^*_\wp$. Moreover, for any $u_0 \in \mathfrak{M}^*_\wp$, the sequence $\{\Gamma^n u_0\}$ $\wp-$converges to w.*

Proof. Let u_0 be an element of $\mathfrak{M}^*_\wp$ and define a sequence $\{u_n\}$ by,

$$u_n = \Gamma^n u_0, \qquad \text{for all } n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, let $\mu_n = \wp_\lambda (u_n, u_{n+1})$. Assume $\mu_n > 0$ for all $n \in \mathbb{N}$. From (31), we have

$$(1-p)\,\mho(\mu_n) < p\mho(\mu_{n-1}), \qquad \text{for all } n \in \mathbb{N}.$$

Since $p \in \left[0, \frac{1}{2}\right)$, it follows that $p < 1-p$, which implies,

$$p\mho(\mu_n) < p\mho(\mu_{n-1}), \qquad \text{for all } n \in \mathbb{N}.$$

Because $\mho$ is strictly increasing, it follows that,

$$\mu_n < \mu_{n-1}, \qquad \text{for all } n \in \mathbb{N}.$$

Thus, $\{\mu_n\}$ is a decreasing sequence of positive numbers. Therefore, there exists a limit $\mu > 0$ such that,

$$\lim_{n\to\infty} \mu_n = \mu.$$

Next, note that for all $n, m \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}\mho(\wp_\lambda(u_m, u_n)) &\le p\left\{\mho\left(\wp_\lambda\left(\Gamma^{m-1}u_0, \Gamma^m u_0\right)\right) + \mho\left(\wp_\lambda\left(\Gamma^{n-1}u_0, \Gamma^n u_0\right)\right)\right\}\\ &= p\left\{\mho(\lambda_{m-1}) + \mho(\lambda_{n-1})\right\}.\end{aligned}$$

By the continuity of $\mho$, as $n, m \to \infty$, it follows that,

$$\mho(\wp_\lambda(u_m, u_n)) \le p\{\mho(\mu) + \mho(\mu)\} = 2p\mho(\mu).$$

Since $p < \frac{1}{2}$, we have

$$\mho(\wp_\lambda(u_m, u_n)) < \mho(\mu), \qquad \text{for all } m, n \in \mathbb{N}.$$

Using the strictly increasing property of $\mho$, this implies,

$$\wp_\lambda(u_m, u_n) \le \mu, \qquad \text{for all } m, n \in \mathbb{N}.$$

In other words, $\{u_n\}$ is $\wp-$bounded sequence. Since $\mathfrak{M}^*_\wp$ is $\wp-$boundedly compact, there exists a subsequence $\{u_{n_k}\}$ that $\wp-$converges. That is, there exist $w \in \mathfrak{M}^*_\wp$ such that,

$$\wp_\lambda(u_{n_k}, w) \to 0, \qquad \text{as } k \to \infty.$$

Using the continuity of Γ, we have

$$\wp_\lambda(\Gamma u_{n_k}, \Gamma w) \to 0, \qquad \text{as } k \to \infty.$$

Since $\wp$ satisfies the Δ_2-condition, it follows that,

$$\wp_{\frac{\lambda}{2}}\left(u_{n_k}, w\right) \to 0, \qquad \text{and} \qquad \wp_{\frac{\lambda}{2}}\left(\Gamma u_{n_k}, \Gamma w\right) \to 0, \qquad \text{as } k \to \infty.$$

Consequently, we obtain

$$\begin{aligned}\wp_{\lambda}\left(\Gamma w, w\right) &\leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}\left(\Gamma w, \Gamma u_{n_k}\right) + \wp_{\frac{\lambda}{2}}\left(\Gamma u_{n_k}, w\right)\right] \\ &\leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}\left(\Gamma w, \Gamma u_{n_k}\right) + \wp_{\frac{\lambda}{2}}\left(u_{n_k+1}, w\right)\right] \to 0, \qquad \text{as } k \to \infty.\end{aligned}$$

This implies that $\wp_{\lambda}\left(\Gamma w, w\right) = 0$. Furthermore, since $\wp$ satisfies the Δ_2-condition, we conclude that $\Gamma w = w$. In other words, w is a FP of Γ. By analogy with the proof of the uniqueness of the FP in Theorem 4.5, it follows that w is the unique FP of Γ. □

Garai et al. [15] introduced the concept of $\Gamma-$orbitally compact metric spaces and established a FP theorem for such spaces. Subsequently, Haokip and Goswami [17] extended the definition of $\Gamma-$orbitally compact to $b-$metric spaces. Based on these ideas, the concept of $\Gamma-$orbitally compactness can be further extended as follows;

Definition 4.1. *Let $\mathfrak{M}^*_{\wp}$ be a MbMS and let $\Gamma : \mathfrak{M}^*_{\wp} \to \mathfrak{M}^*_{\wp}$ be a mapping. The orbit of Γ is defined as the set,*

$$O_u\left(\Gamma\right) = \{u, \Gamma u, \Gamma^2 u, \ldots\}.$$

The space $\mathfrak{M}^*_{\wp}$ is said to be $\Gamma-$orbitally compact if for every $u \in \mathfrak{M}^*_{\wp}$, each sequence in $O_u\left(\Gamma\right)$ has a $\wp-$convergent subsequence.

Now, we give theorems where the MbMS $\mathfrak{M}^*_{\wp}$ is $\Gamma-$orbitally compact, based on the work in [17], as follows;

Theorem 4.8. *Let $\mathfrak{M}^*_{\wp}$ be a $\Gamma-$orbitally compact and $\wp-$bounded MbMS, where $\wp$ satisfies the Δ_2-type condition. Assume that the set $\{K_\lambda | \lambda > 0\}$, where K_λ satisfies Inequality (6), is bounded. If $\Gamma : \mathfrak{M}^*_{\wp} \to \mathfrak{M}^*_{\wp}$ is a mapping satisfying (31) with $p \in \left[0, \frac{1}{2}\right)$ and $\mathfrak{s}p \in [0, 1)$, then Γ has a unique FP $w \in \mathfrak{M}^*_{\wp}$. Moreover, for all $u \in \mathfrak{M}^*_{\wp}$, we have $\wp_{\lambda}\left(\Gamma^n u, w\right) \to 0$ as $n \to \infty$.*

Proof. Let u be an element of $\mathfrak{M}^*_{\wp}$. Consider the sequence $\{u_n\}$ defined by,

$$u_n = \Gamma^n u, \qquad \text{for all } n \in \mathbb{N}.$$

Next, let $\mu_n = \wp_{\lambda}\left(u_n, u_{n+1}\right)$. Analogous to the procedure in Theorem 4.7, we get that $\{\mu_n\}$ is a decreasing sequence of nonnegative real numbers. Therefore, there exists a limit $\mu \in [0, \infty)$ such that,

$$\mu_n \to \mu, \qquad \text{as } n \to \infty. \tag{36}$$

On the other hand, by using the $\Gamma-$orbital compactness of $\mathfrak{M}^*_{\wp}$, the given sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ that is $\wp-$convergent. That is, there exists $w \in \mathfrak{M}^*_{\wp}$ such that,

$$\wp_{\lambda}\left(u_{n_k}, w\right) \to 0, \qquad \text{as } k \to \infty. \tag{37}$$

Furthermore, since $\wp$ satisfies the Δ_2-condition, we have

$$\wp_{\frac{\lambda}{2}}\left(u_{n_k}, w\right) \to 0, \qquad \text{as } k \to \infty.$$

As a result, it follows that,

$$\wp_\lambda(u_{n_k}, u_{n_k+1}) \leq \mathfrak{s}[\wp_{\frac{\lambda}{2}}(u_{n_k}, w) + \wp_{\frac{\lambda}{2}}(w, u_{n_k+1})] \to 0, \qquad \text{as } k \to \infty.$$

Thus, we obtain

$$\mu_{n_k} \to 0, \qquad \text{as } k \to \infty. \tag{38}$$

From (36) and (38), we conclude that $\mu = 0$, which implies,

$$\wp_\lambda(u_n, u_{n+1}) \to 0, \qquad \text{as } n \to \infty. \tag{39}$$

or equivalently, $\mu_n \to 0$ as $n \to \infty$. Furthermore, we have

$$\mho(\mu_n) \to 0, \qquad \text{as } n \to \infty.$$

Next, using Inequality (31), we obtain

$$\begin{aligned}\mho(\wp_\lambda(u_n, u_m)) &\leq p\{\mho\left(\wp_\lambda\left(\varGamma^{n-1}u, \varGamma^n u\right)\right) + \mho\left(\wp_\lambda\left(\varGamma^{m-1}u, \varGamma^m u\right)\right)\} \\ &= p\{\mho(\mu_{n-1}) + \mho(\mu_{m-1})\}.\end{aligned}$$

As a result, using the continuity of $\mho$ and $\mho(t) = 0 \Leftrightarrow t = 0$, we have

$$\mho(\wp_\lambda(u_n, u_m)) \to 0, \qquad \text{as } n, m \to \infty,$$

which implies,

$$\wp_\lambda(u_n, u_m) \to 0, \qquad \text{as } n, m \to \infty.$$

Thus, $\{u_n\}$ is $\wp-$CS. Furthermore, it follows that,

$$\wp_\lambda(u_n, w) \leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(u_n, u_{n_k}) + \wp_{\frac{\lambda}{2}}(u_{n_k}, w)\right] \to 0, \qquad \text{as } n \to \infty. \tag{40}$$

This shows that $\{u_n\}$ $\wp-$converges to w. Note that for all $n \in \mathbb{N}$, we have

$$\wp_\lambda(w, \varGamma w) \leq \mathfrak{s}\left[\wp_{\frac{\lambda}{2}}(w, u_{n+1}) + \wp_{\frac{\lambda}{2}}(u_{n+1}, \varGamma w)\right],$$

which leads to

$$\begin{aligned}\mho(\wp_\lambda(w, \varGamma w)) &\leq \mho\left(\mathfrak{s}[\wp_{\frac{\lambda}{2}}(w, u_{n+1}) + \wp_{\frac{\lambda}{2}}(u_{n+1}, \varGamma w)]\right) \\ &\leq \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(w, u_{n+1})\right) + \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}\left(\varGamma^{n+1}u_0, \varGamma w\right)\right) \\ &\leq \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(w, u_{n+1})\right) + \mathfrak{s}p[\mho\left(\wp_{\frac{\lambda}{2}}\left(\varGamma^n u_0, \varGamma^{n+1}u_0\right)\right) + \mho\left(\wp_{\frac{\lambda}{2}}(w, \varGamma w)\right)] \\ &= \mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}(w, u_{n+1})\right) + \mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}(u_n, u_{n+1})\right) + \mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}(w, \varGamma w)\right).\end{aligned}$$

Since $\wp$ satisfies the Δ_2-condition and $\wp_\lambda(u_n, w) \to 0$, and $\wp_\lambda(u_n, u_m) \to 0$ as $n, m \to \infty$, we have

$$\wp_{\frac{\lambda}{2}}(u_n, w) \to 0 \qquad \text{and} \qquad \wp_{\frac{\lambda}{2}}(u_n, u_m) \to 0, \qquad \text{as } n, m \to \infty.$$

Furthermore, since $\mho$ is continuous, it follows that,

$$\mho\left(\wp_{\frac{\lambda}{2}}(u_n, w)\right) \to 0 \qquad \text{and} \qquad \mho\left(\wp_{\frac{\lambda}{2}}(u_n, u_m)\right) \to 0, \qquad \text{as } n, m \to \infty.$$

As a result, we obtain

$$\mho\left(\wp_{\lambda}\left(w,\Gamma w\right)\right)-\mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}\left(w,\Gamma w\right)\right)\leq\mathfrak{s}\mho\left(\wp_{\frac{\lambda}{2}}\left(w,u_{n+1}\right)\right)+\mathfrak{s}p\mho\left(\wp_{\frac{\lambda}{2}}\left(u_{n},u_{n+1}\right)\right)\to 0,\quad\text{as } n\to\infty,$$

which implies,

$$\mho\left(\wp_{\lambda}\left(w,\Gamma w\right)\right)\leq\mathfrak{s}p\,\mho\left(\wp_{\frac{\lambda}{2}}\left(w,\Gamma w\right)\right).$$

Using an argument analogous to the proof of (29), there exists $K>0$ such that,

$$\mho\left(\wp_{\lambda}\left(w,\Gamma w\right)\right)\leq(\mathfrak{s}p)^{n}\,\mho\left(\wp_{\frac{\lambda}{2^{n}}}\left(w,\Gamma w\right)\right)\leq(\mathfrak{s}p)^{n}K,\qquad\text{for all } n\in\mathbb{N}.$$

Since $\mathfrak{s}p\in[0,1)$, this implies,

$$\mho\left(\wp_{\lambda}\left(w,\Gamma w\right)\right)\to 0,\qquad\text{as } n\to\infty.$$

Therefore, we conclude that $\mho\left(\wp_{\lambda}\left(w,\Gamma w\right)\right)=0$. Since $\mho\left(u\right)=0\Leftrightarrow u=0$, we have $\wp_{\lambda}\left(w,\Gamma w\right)=0$. Based on the Δ_2-condition property of $\wp$, we get $\Gamma w=w$. Hence, w is a FP of Γ. Finally, by analogy with the proof of the uniqueness of the FP in Theorem 4.5, we conclude that w is the unique FP of Γ. □

In the previous discussion, Theorem 4.5 does not hold for $p\geq\frac{1}{2}$. Haokip and Goswami [17] provided a theorem in $b-$metric space, where they extended the bound on p by assuming that Γ is asymptotically regular. The concept of asymptotically regular was introduced by Browder [6]. Now, we define the concept of an asymptotically regular mapping in $\mathfrak{M}^{*}_{\wp}$ as follows.

Definition 4.2. *Let $\mathfrak{M}^{*}_{\wp}$ be a MbMS. A mapping $\Gamma:\mathfrak{M}^{*}_{\wp}\to\mathfrak{M}^{*}_{\wp}$ is said to be asymptotically regular if for all $u\in\mathfrak{M}^{*}_{\wp}$,*

$$\wp_{\lambda}\left(\Gamma^{n}u,\Gamma^{n+1}u\right)\to 0,\qquad\textit{as } n\to\infty.$$

The following theorem is based on the work of Haokip and Goswami [17], where the mapping $\Gamma:\mathfrak{M}^{*}_{\wp}\to\mathfrak{M}^{*}_{\wp}$ is asymptotically regular in a complete and bounded MbMS $\mathfrak{M}^{*}_{\wp}$.

Theorem 4.9. *Let $\mathfrak{M}^{*}_{\wp}$ be a $\wp-$complete and $\wp-$bounded MbMS such that $\wp$ satisfies the Δ_2-type condition. Assume that the set $\{K_{\lambda}|\lambda>0\}$, where K_{λ} satisfies Inequality (6), is bounded. If $\Gamma:\mathfrak{M}^{*}_{\wp}\to\mathfrak{M}^{*}_{\wp}$ is an asymptotically regular mapping satisfying (31) with $p>0$ and $sp\in[0,1)$, then Γ has a unique FP.*

Proof. Let u be an element in $\mathfrak{M}^{*}_{\wp}$. Consider the sequence $\{u_n\}$ defined by,

$$u_{n}=\Gamma^{n}u,\qquad\text{for all } n\in\mathbb{N}.$$

Based on Inequality (31), we obtain

$$\mho\left(\wp_{\lambda}\left(\Gamma^{n+1}u,\Gamma^{m+1}u\right)\right)\leq p[\mho\left(\wp_{\lambda}\left(\Gamma^{n}u,\Gamma^{n+1}u\right)\right)+\mho\left(\wp_{\lambda}\left(\Gamma^{m}u,\Gamma^{m+1}u\right)\right)].$$

Since Γ is asymptotically regular, we have

$$\mho\left(\wp_{\lambda}\left(\Gamma^{n+1}u,\Gamma^{m+1}u\right)\right)\to 0,\qquad\text{as } n,m\to\infty.$$

Moreover, since $\mho$ satisfies conditions (i) and (iii) in Definition 2.2, it follows that,

$$\wp_{\lambda}\left(\Gamma^{n+1}u,\Gamma^{m+1}u\right)\to 0,\qquad\text{as } n,m\to\infty.$$

In other words, $\{u_n\}$ is a $\wp-$CS. By the completeness of $\mathfrak{M}^{*}_{\wp}$, there exists a point $w\in\mathfrak{M}^{*}_{\wp}$ such that,

$$\wp_{\lambda}\left(u_{n},w\right)\to 0,\qquad\text{as } n\to\infty.$$

The next step of proving that w is the unique FP of Γ follows analogously from the argument in Theorem 4.8. □

5 Discussion

In this paper, the definitions of convergent and Cauchy sequence differ from those presented in [13]. Consequently, we require the MbM to satisfy either the Δ_2-type condition or the Δ_2-condition. This distinction aims to provide a deeper understanding of the topology in MbMS, particularly regarding the properties of closed sets in complete MbMS. Moreover, throughout the proof process, some inequalities can be satisfied using the Δ_2-type condition. Therefore, we encourage readers to explore the application of the theorems with this understanding in mind.

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References

[1] R. P. Agarwal, M. Meehan, & D. O'Regan (2001). *Fixed Point Theory and Applications*. Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511543005.

[2] A. Ali, M. Arshad, A. Hussain, N. Hussain & S. M. Alsulami (2022). On new generalized θ_b-contractions andrelated fixed point theorems. *Journal of Inequalities and Applications*, *2022*, Article ID: 37. https://doi.org/10.1186/s13660-022-02770-8.

[3] S. Anwar, M. Nazam, H. H. A. Sulami, A. Hussain, K. Javed & M. Arshad (2022). Existence fixed-point theorems in the partial $b-$metric spaces and an application to the boundary value problem. *AIMS Mathematics*, *7*(5), 8188–8205. https://doi.org/10.3934/math.2022456.

[4] S. Banach (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, *3*(1), 133–181. http://eudml.org/doc/213289.

[5] M. Boriceanu (2009). Fixed point theory on spaces with vector–valued $b-$metrics. *Demonstratio Mathematica*, *XLII*(4), 825–836. https://doi.org/10.1515/dema-2009-0415.

[6] F. E. Browder & W. V. Petryshyn (1966). The solution by iteration of nonlinear functional equations in Banach spaces. *Bulletin of The American Mathematical Society*, *72*(3), 571–575.

[7] A. Buyukkaya & M. Ozturk (2022). Some common fixed point results for Z_E-contractions in modular $b-$metric spaces. *Topological Algebra and its Applications*, *10*(1), 196–215. https://doi.org/10.1515/taa-2022-0127.

[8] K. Chaira, A. Eladraoui, M. Kabil & A. Kamouss (2021). Common fixed point for some generalized contractive mappings in a modular metric space with a graph. *Fixed Point Theory and Algorithms for Sciences and Engineering*, *2021*, Article ID: 4. https://doi.org/10.1186/s13663-021-00690-8.

[9] V. V. Chistyakov (2010). Modular metric spaces, I : Basic concepts. *Nonlinear Analysis: Theory, Methods & Apllications*, *72*(1), 1–14. https://doi.org/10.1016/j.na.2009.04.057.

[10] V. V. Chistyakov (2015). *Metric Modular Spaces: Theory and Applications*. SpringerBriefs in Mathematics. Springer, Cham. https://doi.org/10.1007/978-3-319-25283-4.

[11] P. Corazza (1999). Introduction to metric-preserving functions. *The American Mathematical Monthly*, *106*(4), 309–232. https://doi.org/10.2307/2589554.

[12] S. Czerwik (1993). Contraction mappings in $b-$metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis*, *1*(1), 5–11. http://dml.cz/dmlcz/120469.

[13] M. E. Ege & C. Alaca (2018). Some results for modular $b-$metric spaces and an application to system of linear equations. *Azerbaijan Journal of Mathematics*, *8*(1), 3–14. https://www.azjm.org/volumes/0801/0801-1.pdf.

[14] H. Faraji & K. Nourozi (2017). A generalization of Kannan and Chatterjea fixed point theorems on complete $b-$metric space. *Sahand Communications in Mathematical Analysis*, *6*(1), 77–86. https://doi.org/10.22130/scma.2017.23831.

[15] H. Garai, T. Senapati & L. K. Dey. A study on Kannan type contractive mappings. arXiv preprint arXiv:1707.06383 2017. https://doi.org/10.48550/arXiv.1707.06383.

[16] J. Górnicki (2017). Fixed point theorems for Kannan type mappings. *Journal of Fixed Point Theory and Applications*, *19*(1), 2145–2152. https://doi.org/10.1007/s11784-017-0402-8.

[17] N. Haokip & N. Goswami (2019). Some fixed point theorems for generalized Kannan type mappings in $b-$metric spaces. *Proyeccioness Journal of Mathematics*, *38*(4), 763–782. https://doi.org/10.22199/issn.0717-6279-2019-04-0050.

[18] A. Hayati, L. Harini, A. Winarni & N. Muhassanah (2022). Teori titik tetap untuk pemetaan $(\psi, \varphi)_{\Omega}-$kontraksi pada ruang $p-$metrik modular berorder. *Pythagoras: Jurnal Matematika dan Pendidikan Matematika*, *17*(2), 400–414. https://doi.org/10.21831/pythagoras.v17i2.52985.

[19] A. Hayati, N. Sofiyati & D. L. Kartika (2023). Teori titik tetap untuk pemetaan tipe Kannan yang diperumum dalam ruang $b-$metrik modular lengkap. *Jambura Journal of Mathematics*, *5*(2), 363–379. https://https://doi.org/10.34312/jjom.v5i2.20571.

[20] M. Iqbal, A. Batool, A. Hussain & H. A. Sulami (2024). Generalized Wardowski type contractive mappings in $b-$metric spaces and some fixed point results with applications in optimization problem and modeling biological ecosystem. *PLOS One*, *19*(12), Article ID: e0313033. https://doi.org/10.1371/journal.pone.0313033.

[21] R. Kannan (1969). Some result on fixed points–II. *The American Mathematical Monthly*, *76*(4), 405–408. https://doi.org/10.2307/2316437.

[22] H. Long-Guang & Z. Xian (2007). Cone metric spaces and fixed point theorems of contractive mappings. *Journal of Mathematical Analysis and Applications*, *332*(2), 1468–1476. https://doi.org/10.1016/j.jmaa.2005.03.087.

[23] M. Mudhesh, A. Hussain, M. Arshad & H. Alsulami (2023). A contemporary approach of integral Khan-type multivalued contractions with generalized dynamic process and an application. *Mathematics*, *11*(20), Article ID: 4318. https://doi.org/10.3390/math11204318.

[24] J. Musielak & W. Orlicz (1959). On modular space. *Studia Mathematica*, *18*, 49–65. https://doi.org/10.4064/sm-18-1-49-65.

[25] M. Nazam, Z. Hamid, H. A. Sulami & A. Hussain (2021). Common fixed-point theorems in the partial $b-$metric spaces and an application to the system of boundary value problems. *Journal of Function Spaces*, *2021*(1), Article ID: 7777754. https://doi.org/10.1155/2021/7777754.

[26] M. Nazam, A. Hussain & A. Asiri (2023). On a common fixed point theorem in vector-valued $b-$metric spaces: Its consequences and application. *AIMS Mathematics*, *8*(11), 26021–26044. https://doi.org/10.3934/math.20231326.

[27] M. Ozturk, F. Golkarmanesh, A. Buyukkaya & V. Parvaneh (2023). Generalized almost simulative $\hat{Z}^{\theta}_{\psi^*}-$contraction mappings in modular $b-$metric spaces. *Journal of Mathematical Extension*, *17*(2), 1–37. https://doi.org/10.30495/JME.2023.2424.

[28] V. Parvaneh, N. Hussain, M. Khorshidi, N. Mlaiki & H. Aydi (2019). Fixed point results for generalized $\mathcal{F}-$contractions in modular $b-$metric spaces with applications. *Mathematics*, *7*(10), Article ID: 887. https://doi.org/10.3390/math7100887.

[29] S. Reich (1971). Some remarks concerning contraction mappings. *Canadian Mathematical Bulletin*, *14*(1), 121–124. https://doi.org/10.4153/CMB-1971-024-9.

[30] N. Saleem, M. Abbas & K. Sohail (2021). Approximate fixed point results for $(\alpha - \mu)-$type and $(\beta - \psi)-$type fuzzy contractive mappings in b fuzzy metric spaces. *Malaysian Journal of Mathematical Sciences*, *15*(2), 267–281.

[31] S. Sedghi & N. Shobe (2012). Common fixed point theorems in $b-$fuzzy metric spaces. *Nonlinear Functional Analysis and Applications*, *17*(3), 349–359.

[32] F. Shaddad, M. S. M. Noorani & S. M. Alsulami (2014). Fixed point results in cone metric spaces for multivalued maps. *Malaysian Journal of Mathematical Sciences*, *8*(S), 83–102.

[33] S. Shukla (2014). Partial $b-$metric spaces and fixed point theorems. *Mediterranean Journal of Mathematics*, *11*, 703–711. https://doi.org/10.1007/s00009-013-0327-4.

[34] S. G. Tanha, A. Bodaghi & A. N. Motlagh (2022). Some results in metric modular spaces. *International Journal of Nonlinear Anlysis and Applications*, *13*(2), 983–988. https://doi.org/10.22075/ijnaa.2022.25676.3091.

[35] D. Turkoglu & N. Manav (2018). Fixed point theorems in a new type of modular metric spaces. *Fixed Point Theory and Applications*, *2018*, Article ID: 25. https://doi.org/10.1186/s13663-018-0650-3.

[36] P. Tyagi, S. S. Chauhan, N. Mani & R. Shukla (2024). $\mathcal{F}-$modular $b-$metric spaces and some analogies of classical fixed point theorems. *International Journal of Analysis and Applications*, *22*, Article ID: 66. https://doi.org/10.28924/2291-8639-22-2024-66.